ISRAEL JOURNAL OF MATHEMATICS, Vol. 29, No. 4, 1978

QUOTIENTS OF $C[0,1]$ WITH SEPARABLE DUAL

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ABSTRACT

A necessary and sufficient condition for an operator from $C(K)$, K compact metric, into a Banach space X to be an isomorphism on a subspace of $C(K)$ isometric to $C_0(\omega^\omega)$ is given.

0. Introduction

In the papers of Pelczynski [7] and Rosenthal [10] operators from $C(K)$, K compact metric, into a Banach space X were considered. Pelczynski showed that a non-weakly compact operator with domain $C(K)$ is an isomorphism on a subspace Y of $C(K)$ with Y isometric to c_0. In his paper Rosenthal proved that if K is a compact metric space and $T^*B_{X^*}$ is non-separable then there is a subspace Y of $C(K)$ with Y isometric to $C[0,1]$ such that T restricted to Y is an isomorphism. Both of these results use a condition on $T^*B_{X^*}$ to produce the required subspace Y. In this paper we give a condition on $T^*B_{X^*}$ which ensures that there is a subspace Y of $C(K)$, Y isometric to $C_0(\omega^\omega)$, with T an isomorphism on Y.

To state our result we need to recall the definition of index as introduced by Szlenk [12]. Let A be a bounded subset of a separable Banach space X and B a bounded subset of X^*. For any $\varepsilon > 0$ let $P_0(\varepsilon, A, B) = B$ and having defined $P_\alpha(\varepsilon, A, B)$ for any ordinal α we let $P_{\alpha+1}(\varepsilon, A, B) = \{b \mid \text{there are a sequence } (b_n)_{n=1}^\infty \subset P_\alpha(\varepsilon, A, B) \text{ and a sequence } (a_n)_{n=1}^\infty \subset A \text{ such that } b_n \xrightarrow{w^*} b,\ a_n \xrightarrow{w} 0, \text{ and } \overline{\lim}_n \langle b_n, a_n\rangle \geqq \varepsilon\}$. If β is a limit ordinal, we let $P_\beta(\varepsilon, A, B) = \bigcap_{\alpha<\beta} P_\alpha(\varepsilon, A, B)$. Finally we define the ε-index as

† This is part of the author's Ph.D. dissertation being prepared at The Ohio State University under the supervision of Professor W. B. Johnson. This research was supported in part by NSF grant MPS 72-04634-A03 and a University Fellowship of the Ohio State University.

Received January 18, 1977

$$\eta(\varepsilon, A, B) = \sup\{\alpha < \omega_1 \mid P_\alpha(\varepsilon, A, B) \neq \emptyset\}.$$

Under the additional assumptions that X^* be separable and B be w^*-closed Szlenk shows that the sets $P_\alpha(\varepsilon, A, B)$ are w^*-closed and $\eta(\varepsilon, A, B) < \omega_1$ for each $\varepsilon > 0$. We do not wish to restrict to the case of X^* separable since for us X may be $C[0,1]$. Actually, if B is separable, and $B = \bar{B}^{w^*}$ the argument of Szlenk still shows that $\eta(\varepsilon, A, B) < \omega_1$. Indeed, if we alter the definition of the sets $P_\alpha(\varepsilon, A, B)$ so that rather than requiring the sequence $(a_n)_{n=1}^\infty$ to converge weakly to zero that it converge to zero with respect to B, i.e., $\langle b, a_n\rangle \to 0$ for all $b \in B$ and we let $Q_\alpha(\varepsilon, A, B)$ be the corresponding sets, then for each $\alpha < \omega_1$, $Q_\alpha(\varepsilon, A, B) \supset P_\alpha(\varepsilon, A, B)$. The sets $Q_\alpha(\varepsilon, A, B)$ are w^*-closed, and the argument of Szlenk shows that $\sup\{\alpha : Q_\alpha(\varepsilon, A, B) \neq \emptyset\} < \omega_1$. Consequently $\eta(\varepsilon, A, B) < \omega_1$.

Another property of the sets $P_\alpha(\varepsilon, A, B)$ which Szlenk states is the following

Proposition 0.1. *If X and Y are separable Banach spaces and T is an isomorphism from X into Y, then $P_\alpha(\varepsilon, A, B) = T^*(P_\alpha(\varepsilon, TA, T^{*-1}B))$ for every $\alpha < \omega_1$ and $\varepsilon > 0$.*

Proof. Clearly $P_0(\varepsilon, A, B) = T^*(P_0(\varepsilon, TA, T^{*-1}B))$. Inductively assume that for all $\beta < \alpha$, $P_\beta(\varepsilon, A, B) = T^*(P_\beta(\varepsilon, TA, T^{*-1}B))$. For limit ordinals α it is obvious that this implies that $P_\alpha(\varepsilon, A, B) = T^*(P_\beta(\varepsilon, TA, T^{*-1}B))$.

If $\alpha = \beta + 1$ for some β and $b \in P_{\beta+1}(\varepsilon, A, B)$ then there exist a sequence $(b_n)_{n=1}^\infty \subset P_\beta(\varepsilon, A, B)$ and a sequence $(a_n)_{n=1}^\infty \subset A$ such that $b_n \xrightarrow{w^*} b$, $a_n \xrightarrow{w} 0$, and $\overline{\lim}_n \langle b_n, a_n\rangle \geqq \varepsilon$. By passing to a subsequence if necessary we can find a sequence $(c_n)_{n=1}^\infty \subset P_\beta(\varepsilon, TA, T^{*-1}B)$ such that $T^*c_n = b_n$ for each n and $(c_n)_{n=1}^\infty$ converges to an element c of $\overline{T^{*-1}B}^{w^*}$. Thus $\overline{\lim}_n \langle c_n, Ta_n\rangle = \overline{\lim}_n \langle b_n, a_n\rangle \geqq \varepsilon$ and therefore $c \in P_{\beta+1}(\varepsilon, TA, T^{*-1}B)$ and $T^*c = w^* \lim_n b_n = b$.

Conversely, if $c \in P_{\beta+1}(\varepsilon, TA, T^{*-1}B)$ there exist a sequence $(c_n)_{n=1}^\infty \subset P_\beta(\varepsilon, TA, T^{*-1}B)$ and a sequence $(d_n)_{n=1}^\infty \subset TA$ such that $c_n \xrightarrow{w^*} c$, $d_n \xrightarrow{w} 0$, and $\overline{\lim}_n \langle c_n, d_n\rangle \geqq \varepsilon$. Since T is an isomorphism, $T^{-1}d_n \xrightarrow{w} 0$. Clearly $\overline{\lim}_n \langle T^*c_n, T^{-1}d_n\rangle = \overline{\lim}_n \langle c_n, d_n\rangle \geqq \varepsilon$ and thus $T^*c \in P_{\beta+1}(\varepsilon, A, B)$.

We can now state our result on $C(\omega^\omega)$.

Theorem 0.2. *Let K be a compact metric space and let T be a bounded linear operator from $C(K)$ into a Banach space X. If there is an $\varepsilon > 0$ such that*

$$\eta(\varepsilon, B_{C(K)}, T^*B_{X^*}) \geqq \omega$$

then there is a subspace Y of $C(K)$ such that Y is isometric to $C_0(\omega^\omega)$ and T/Y is an isomorphism.

The condition $\eta(\varepsilon, B_{C(K)}, T^*B_{X^*}) \geqq \omega$ for some $\varepsilon > 0$ is also a necessary condition since $\overline{T(C(K))}$ is separable. This is a consequence of Proposition 0.1 and

LEMMA 0.3. *If* $\beta < \omega_1$, *then for all* $\alpha < \omega_1$, $P_\alpha(1, B_{C(\beta)}, B_{C(\beta)^*}) = \overline{\text{co}}\{\pm\delta_\gamma : \gamma \in [1,\beta]^{(\alpha)}\}$ *where* δ_γ *is the point mass measure at* γ *and* $[1,\beta]^{(\alpha)}$ *is the* α*th derived set of* $[1,\beta]$.

PROOF. Clearly the set $P_\alpha(1, B_{C(\beta)}, B_{C(\beta)^*})$ is convex for each $\alpha < \omega_1$ and $P_0(1, B_{C(\beta)}, B_{C(\beta)^*}) = \overline{\text{co}}\{\pm\delta_\gamma : \gamma \in [1,\beta]\}$. Suppose the lemma is true for all $\rho < \alpha$. If α is a limit ordinal, the induction step is trivial. If $\alpha = \rho + 1$ for some ρ and $\gamma \in [1,\beta]^{(\rho+1)}$, there is a sequence $(\gamma_n)_{n=1}^\infty \subset [1,\beta]^{(\rho)}$ with $\gamma_n \uparrow \gamma$. Since $\delta_{\gamma_n} \xrightarrow{w^*} \delta_\gamma$, $1_{[\gamma_{n-1}+1,\gamma_n]} \xrightarrow{w} 0$, and

$$\lim_n \langle \delta_{\gamma_n}, 1_{[\gamma_{n-1}+1,\gamma_n]}\rangle = 1, \quad \delta_\gamma \in P_{\rho+1}(1, B_{C(\beta)}, B_{C(\beta)^*}).$$

On the other hand, if $\mu \in P_{\rho+1}(1, B_{C(\beta)}, B_{C(\beta)^*})$, we can assume that

$$\mu = \sum_{i=1}^\infty a_i\delta_{\gamma_i} \quad \text{where} \quad \sum_{i=1}^\infty |a_i| \leqq 1 \quad \text{and} \quad (\gamma_i)_{i=1}^\infty \subset [1,\beta]^{(\rho)},$$

by induction. There exist a sequence $(\mu_n)_{n=1}^\infty \subset P_\rho(1, B_{C(\beta)}, B_{C(\beta)^*}) = \overline{\text{co}}\{\pm\delta_\gamma : \gamma \in [1,\beta]^{(\rho)}\}$ and a sequence $(f_n) \subset B_{C(\beta)}$ such that $\mu_n \xrightarrow{w^*} \mu$, $f_n \xrightarrow{w} 0$, and $\overline{\lim}_n \langle \mu_n, f_n\rangle = 1$. If for some i, $\gamma_i \notin [1,\beta]^{(\rho+1)}$ then since γ_i is an isolated point of $[1,\beta]^{(\rho)}$, $\mu_n(\{\gamma_i\}) \to \mu(\{\gamma_i\})$. But then because $f_n \xrightarrow{w} 0$, $f_n(\gamma_i) \to 0$ and hence $\overline{\lim}_n \langle \mu_n, f_n\rangle = 1 - |a_i| < 1$.

One might conjecture that the theorem may be generalized as follows:

There is a function $\phi : [0,\omega_1) \to [1,\omega_1)$ such that if K is a compact metric space and T is a bounded linear operator from $C(K)$ into a Banach space X such that for some $\varepsilon > 0$

$$\eta(\varepsilon, B_{C(K)}, T^*B_{X^*}) \geqq \phi(\alpha)$$

then there is a subspace Y of $C(K)$ isometric to $C_0(\omega^{\omega^\alpha})$ such that $T_{|Y}$ is an isomorphism.

(We consider only the spaces $C_0(\omega^{\omega^\alpha})$, $\alpha < \omega_1$, since by [2] these form a complete set of representatives for the isomorphism classes of $C(K)$ spaces for K countable.)

A reformulation of the earlier mentioned result of Pelczynski shows that $\phi(0) = 1$ is the correct choice (See Remark 2 following Lemma 1.3). Thus an obvious candidate for $\phi(\alpha)$ is ω^α for each $\alpha < \omega_1$. However, in [1] we show that there is an operator $T : C_0(\omega^{\omega^2}) \xrightarrow{\text{onto}} C_0(\omega^{\omega^2})$ such that

$$\eta(\tfrac{1}{2}, B_{C_0(\omega^{\omega^2})}, T^*B_{C_0(\omega^{\omega^2})^*}) \geqq \omega^2$$

but there is *no* subspace Y of $C_0(\omega^{\omega^2})$ such that Y is isomorphic to $C_0(\omega^{\omega^2})$ and $T_{|Y}$ is an isomorphism.

We use standard Banach space notation as may be found in [6]. All operators between Banach spaces will be bounded and linear. If K is a compact metric space we will identify $C(K)^*$ with the space of all finite signed Borel measures on K. All subsets of such K considered will be Borel measurable.

Besides the notation $K^{(\alpha)}$ for the αth derived set of a topological space K which we used above, it will be convenient to have several specialized notations. $K^{(\alpha)} - K^{(\alpha+1)}$ will be abbreviated to $K^{d(\alpha)}$. If a subset A of K is written $A = \{a_\alpha\}_{\alpha\in\Lambda}$ for some set of ordinals Λ, we mean that the correspondence $a_\alpha \leftrightarrow \alpha$ is a homeomorphism where Λ is given the order topology. If β is an ordinal, $C(\beta)$ (resp. $C_0(\beta)$) denotes the space of real or complex valued continuous functions on the ordinals not greater than β (resp., and vanishing at β). We will give proofs only for the real case but they may be easily adapted to the complex case. The reader may wish to consult [11] for information about ordinals and their arithmetic.

Finally we will have occasion to use the convention that if M is an infinite set of positive integers $\lim_M a_n$ means $\lim_{k\to\infty} a_{n_k}$ where $k \to n_k$ is a strictly increasing map of $\mathbf{N}$ onto M.

1. Some technical lemmas

We begin with a few technical lemmas which we will use in section 2.

For the first lemma we need to define a quantitative measure of the non-uniform absolute continuity of a set of measures.

Let $F \subset B_{C(K)^*}$, K a compact Hausdorff space, and let μ be a probability measure on K such that for all $\nu \in F$, $\nu \ll \mu$ (i.e., $F \subset L_1(\mu)$). Define $\lambda(F,\mu) = \sup\{\varepsilon \mid \forall \delta > 0, \exists \nu \in F$ and $A \subset K$ such that $|\nu|(A) \geqq \varepsilon$ and $\mu(A) < \delta\}$.

Observe that if $\lambda(F,\mu) = \rho$ and B_n is a decreasing sequence of subsets of K such that $\bigcap_{n=1}^\infty B_n = \emptyset$ then for every $\gamma > \rho$ there is an N such that $|\nu|(B_N) \leqq \gamma$ for all $\nu \in F$.

LEMMA 1.1. *Let $F \subset B_{C(K)^*}$, $F \neq \emptyset$, and μ a probability measure on K such that $F \subset L_1(\mu)$. Then there exist a sequence $(\nu_n)_{n=1}^\infty \subset F$ and a sequence of disjoint closed sets $(A_n)_{n=1}^\infty$, $A_n \subset K$ for each n, such that*

$$(\nu_{n|A_n^c})_{n=1}^\infty \text{ is u.a.c. } \mu$$

(uniformly absolutely continuous with respect to μ)

and

$$\lim_{n\to\infty} |\nu_n|(A_n) = \lambda(F,\mu).$$

The proof of this lemma is contained in the proof of theorem 6 of [5], but we will include a proof for completeness.

PROOF. If $\lambda(F,\mu)=0$, the result is obvious, so assume that $\lambda(F,\mu)=\delta>0$. We will choose the measures $(\nu_n)_{n=1}^{\infty}$ by induction.

Let $\nu_1 \in F$ such that there is a subset B_1 of K with $|\nu_1|(B_1) > 3\delta/4$ and $\mu(B_1) < \delta/4$. Next choose $\nu_2 \in F$ such that there is a subset B_2 of K with $|\nu_2|(B_2) > 7\delta/8$, $\mu(B_2) < \delta/8$, and $|\nu_1|(B_2) < \delta/8$.

Suppose we have chosen $(\nu_i)_{i=1}^{k-1}$ and $(B_i)_{i=1}^{k-1}$, choose $\nu_k \in F$ such that there is a subset B_k of K with $|\nu_k|(B_k) > \delta(1-1/2^k)$, $\mu(B_k) < \delta/2^{k+1}$, and $|\nu_i|(B_k) < \delta/2^{k+1}$, $1 \leqq i \leqq k-1$. This is possible since the finite set $(\nu_i)_{i=1}^{k-1}$ is u.a.c. μ.

For each k let $B'_k = B_k - \bigcup_{i=k+1}^{\infty} B_i$ and choose a closed set $A_k \subset B'_k$ such that $|\nu_k|(B'_k - A_k) < \delta/2^k$. We have that

$$|\nu_k|(A_k) > |\nu_k|(B'_k) - \delta/2^k \geqq |\nu_k|(B_k) - \frac{\delta}{2^{k-1}} > \delta\left(1 - \frac{1}{2^{k-2}}\right).$$

Also since the sets $(B'_k)_{k=1}^{\infty}$ are disjoint, the sets $(A_k)_{k=1}^{\infty}$ are disjoint.

If $\lambda((\nu_{n|A_n^c})_{n=1}^{\infty},\mu) = \varepsilon > 0$, for every $k \in \mathbf{N}$ we could find a subset D_k of K and an integer n_k such that $|\nu_{k|A_{n_k}^c}|(D_k) > \varepsilon/2$ and $\mu(D_k) < 1/2^k$. But then

$$|\nu_{n_k}|(A_{n_k} \cup D_k) > \delta(1 - 2^{-n_k+2}) + \varepsilon/2.$$

Since $\mu(A_{n_k} \cup D_k) \to 0$, $\lambda((\nu_{n_k})_{k=1}^{\infty},\mu) \geqq \delta + \varepsilon/2$, a contradiction.

The next lemma shows us that we can replace the closed sets in Lemma 1.1 by open sets by making a small sacrifice. This lemma is due to A. Pelczynski (lemma 1 of [8]) in a slightly different form.

LEMMA 1.2. *Let $(\mu_n)_{n=1}^{\infty} \subset B_{C(K)^*}$, K a compact Hausdorff space. Suppose $(F_n)_{n=1}^{\infty}$ is a sequence of disjoint closed subsets of K such that $|\mu_n|(F_n) = |\mu_n|(K)$. Then for every $\varepsilon>0$ there exist a subsequence $(\mu_{n_k})_{k=1}^{\infty}$ of $(\mu_n)_{n=1}^{\infty}$ and disjoint open sets G_k such that*

a) $$|\mu_{n_k}|(G_k) > \|\mu_{n_k}\| - \varepsilon \quad \forall k,$$

b) $$|\mu_{n_l}| \bigcup_{k\neq l} G_k < \varepsilon \quad \forall l.$$

PROOF. We choose the measures and open sets inductively.

Let $(l_i)_{i=1}^{\infty}$ be an increasing sequence of positive integers such that $\Sigma_{i=1}^{\infty} l_i^{-1} < \varepsilon$ and consider $F_1, F_2, \cdots, F_{l_1}$. Since K is a compact Hausdorff space we can find disjoint open sets $G(1,1), G(1,2), \cdots, G(1,l_1)$ such that $G(1,j) \supset F_j$, for $j = 1,2,\cdots,l_1$. For some n_1 there is an infinite set $N_1 \subset \mathbf{N} - \{1,2,\cdots,l_1\}$ such that $|\mu_n|(G(1,n_1)) \leqq 1/l_1$ for every $n \in N_1$. Indeed, if no such n_1 exists then we could find an n such that $|\mu_n|(G(1,j)) > 1/l_1$ for $j = 1,2,\cdots,l_1$. But then $\|\mu_n\| \geqq \Sigma_{j=1}^{l_1} |\mu_n|(G(1,j)) > 1$, a contradiction.

Thus we have n_1 and we can choose an open set $G_1 \subset G(1,n_1)$ with $\bar{G}_1 \subset G(1,n_1)$ and $|\mu_{n_1}|(G_1) > |\mu_{n_1}|(G(1,n_1)) - l_1^{-1}$.

Next let $F(1,n) = F_n - G(1,n_1)$ for each $n \in N_1$ and let $n(1,1)$, $n(1,2),\cdots,n(1,l_2)$ be the first l_2 elements of N_1. As before we can find disjoint open sets $G(2,1), G(2,2),\cdots,G(2,l_2)$ such that $G(2,j) \supset F(1,n(1,j))$ and $G(2,j) \cap G_1 = \emptyset$, for $j = 1,2,\cdots,l_2$. Again we can find an infinite set $N_2 \subset N_1 - \{n(1,1), n(1,2),\cdots,n(1,l_2)\}$ and an index $n(1,k)$, $1 \leqq k \leqq l_2$, such that $|\mu_n|(G(2,k)) \leqq 1/l_2$ for all $n \in N_2$. Let $n_2 = n(1,k)$ and choose an open set G_2 such that $\bar{G}_2 \subset G(2,n_2)$ and $|\mu_{n_2}|(G_2) > |\mu_{n_2}|(G(2,n_2)) - l_2^{-1}$.

Continuing in this fashion we can find measures $(\mu_{n_k})_{k=1}^{\infty}$ and disjoint open sets $(G_k)_{k=1}^{\infty}$ such that $|\mu_{n_k}|(G_k) > \|\mu_{n_k}\| - \Sigma_{i=1}^{k} l_i^{-1}$ and $|\mu_{n_k}|(G_j) \leqq l_j^{-1}$ if $k > j$. Thus $|\mu_{n_k}|(G_k) \geqq \|\mu_{n_k}\| - \varepsilon$ for each k and $|\mu_{n_k}|(\bigcup_{j\neq k} G_j) \leqq \Sigma_{j=1}^{k} l_i^{-1} < \varepsilon$.

We will use Lemmas 1.1 and 1.2 in concert as

LEMMA 1.3. *Let F be a nonempty subset of $B_{C(K)^*}$, K a compact Hausdorff space, and let μ be a probability measure on K such that $F \subset L_1(\mu)$. If $\lambda(F,\mu) = \delta > 0$ then for every $\varepsilon > 0$ there exist a sequence $(\mu_n)_{n=1}^{\infty} \subset F$ and a sequence of disjoint open subsets $(G_n)_{n=1}^{\infty}$ of K such that*

a) $$|\mu_n|(G_n) > \delta - \varepsilon \quad \forall n,$$

b) $$|\mu_m| \bigcup_{n\neq m} G_n < \varepsilon \quad \forall m,$$

c) $$\lambda((\mu_{n|G_n^c})_{n=1}^{\infty}, \mu) < \varepsilon.$$

PROOF. By Lemma 1.1 we can find a sequence $(\nu_k)_{k=1}^{\infty} \subset F$ and a sequence of disjoint closed subsets of K, $(A_k)_{k=1}^{\infty}$, such that $\lim_k |\nu_k|(A_k) = \delta$ and $\lambda((\nu_{k|A_k^c})_{k=1}^{\infty}, \mu) = 0$. To get the required sequence $(\mu_n)_{n=1}^{\infty}$ and open sets $(G_n)_{n=1}^{\infty}$ we apply Lemma 1.2 to $(\nu_{k|A_k^c})_{k=1}^{\infty}$ and $(A_k)_{k=1}^{\infty}$.

REMARK 1. Note that if K is 0 dimensional we can use the regularity of the measures (μ_n) to replace the sets (G_n) by clopen sets (H_n) such that

a) $$|\mu_n|(H_n) > \delta - \varepsilon \quad \forall n,$$

b) $$|\mu_m|\left(\bigcup_{n \neq m} H_n\right) < \varepsilon \quad \forall m,$$

c) $$\lambda\left((\mu_{n|H_n^c})_{n=1}^{\infty}, \mu\right) < \varepsilon.$$

REMARK 2. Using Lemma 1.3 we can show that an operator $T: C(K) \to X$ is not weakly compact if and only if for some $\varepsilon > 0$, $\eta(\varepsilon, B_{C(K)}, T^*B_{X^*}) \geqq 1$. Indeed, it is well known that T is weakly compact if and only if T^* is weakly compact. Also $T^*B_{X^*}$ is relatively weakly compact if and only if there is a measure μ on K such that $T^*B_{X^*}$ is uniformly absolutely continuous with respect to μ. Hence suppose $T^*B_{X^*}$ is uniformly absolutely continuous with respect to some measure μ on K. Then if $(\mu_n)_{n=1}^{\infty} \subset T^*B_{X^*}$ and $(f_n)_{n=1}^{\infty} \subset B_{C(K)}$ with $f_n \xrightarrow{w} 0$, we have that $\langle \mu_n, f_n \rangle \to 0$, for, by Egorov's Theorem $f_n \to 0$ uniformly on most of K and the μ_n's are small on the remainder. Consequently, $\eta(\varepsilon, B_{C(K)}, T^*B_{X^*}) = 0$ for every $\varepsilon > 0$. Conversely if no such μ exists, we can find a measure ν and a subset $F \subset T^*B_{X^*}$ such that $F \subset L_1(\nu)$ and $\lambda(F, \nu) = \delta > 0$. Hence by applying Lemma 1.3 we can find a sequence of measures $(\nu_n)_{n=1}^{\infty}$ and a sequence of disjoint open sets $(G_n)_{n=1}^{\infty}$ such that $|\nu_n|(G_n) > 3/4\delta$ and $|\nu_n| \bigcup_{k \neq n} G_k < \delta/4$. We now choose a continuous function $f_n \in B_{C(K)}$ for each n such that $\langle \nu_n, f_n \rangle > \delta/2$ and $\sup f_n \subset G_n$. Clearly $f_n \xrightarrow{w} 0$ and any limit point of $(\nu_n)_{n=1}^{\infty}$ is in $P_1(\delta/2, B_{C(K)}, T^*B_{X^*})$. Therefore $\eta(\delta/2, B_{C(K)}, T^*B_{X^*}) \geqq 1$.

In the proof of Theorem 0.2 we will construct a subset F of $C(K)^*$ and the subspace Y so that the evaluation map taking Y into $C_0(F)$ is an isomorphism onto $C_0(F)$ and F is homeomorphic to $[1, \omega^{\omega}]$. The space Y will be a c_0 sum of spaces Y_n isomorphic to $C(\omega^n)$, $n = 1, 2, \cdots$. Let us examine the structure of $C(\omega^n)$.

If A is a set of ordinals and $a \in A$ let

$$a^- = \begin{cases} \sup\{b \mid b \in A, b < a\} & \text{if } \; a \neq \inf A \\ 0 & \text{if } \; a = \inf A, \end{cases}$$

e.g. if $A = \{\omega^3 j + \omega^2 k \mid j, k \in \mathbf{N}\}$ and $a = \omega^3 j_0 + \omega^2 k_0$ then $a^- = \omega^3 j_0 + \omega^2(k_0 - 1)$.

Note that a^- need not belong to A and that if $a \in A - A^{d(0)}$, i.e., a is not isolated in A, $a^- = a$. (We consider A as a subspace of $[1, \sup A]$ with the order topology. For closed sets A this is the same as the order topology on A.) for each k, $k = 0, 1, \cdots, n$ let

$$\mathcal{F}_k = \{1_{(\alpha^-, \alpha]} \mid \alpha \in [1, \omega^n]^{d(k)}\}$$

and $\mathcal{F} = \bigcup_{k=1}^{n} \mathcal{F}_k$. From the Stone Weierstrauss Theorem it follows that $[\mathcal{F}] = C(\omega^n)$.

The spanning set $\mathcal{F}$ will be our model for constructing the space Y_n. Note that we have a bijection between the points of $[1, \omega^n]$ and the set $\{(\alpha^-, \alpha] \mid \alpha \in [1, \omega^n]^{d(k)}, k = 0, 1, \cdots, n\}$. In constructing the space Y_n we will want to imitate the relationship between the point masses $\{\delta_\alpha : \alpha \leqq \omega^n\}$ and the intervals $\{(\alpha^-, \alpha] \mid \alpha \in [1, \omega^n]^{d(k)}, k = 0, 1, \cdots, n\}$. Our goal will be to find a subset $\{\mu_\alpha\}_{\alpha \leqq \omega^n}$ of $T^* B_{X^*}$ and open sets $\{G_\alpha\}_{\alpha \leqq \omega^n}$ such that if $\alpha \in [1, \omega^n]^{d(k)}$ and $\beta \in (\alpha^-, \alpha]$ then $\lambda(\{\mu_{\beta | G_\alpha^c}\}_{\beta \in (\alpha^-, \alpha)}, \mu)$ is small and $\lambda(\{\mu_\beta\}_{\beta \in (\alpha^-, \alpha]}, \mu)$ is uniformly bounded away from zero.

The next lemma describes a subspace of $C(K)$ isometric to $C(\omega^k)$ and a norming subset of the dual. The measures $\{\mu_\alpha\}_{\alpha \leqq \omega^k}$ and the clopen sets $\{G_\alpha\}_{\alpha \leqq \omega^k}$ described below are the object of our construction in the next section.

LEMMA 1.4. *Let $A = \{\mu_\alpha\}_{\alpha \leqq \omega^k} \subset B_{C(K)^*}$ where K is a compact Hausdorff space and A is considered in the w^* topology. Suppose that there are constants $a \in \mathbf{R}$ and $\varepsilon > 0$ and that for each $\alpha \leqq \omega^k$ there is a non-empty clopen subset of K, G_α, such that*

a) *$G_\alpha \subsetneqq G_\beta$ if $\beta \in [1, \omega^k]^{d(r)}$ for some $r \leqq k$ and $\beta^- < \alpha < \beta$,*
b) *$G_\alpha \cap G_{\alpha'} = \emptyset$ if $\alpha, \alpha' \in [1, \omega^k]^{d(s)}$ for some $s, \alpha \neq \alpha'$,*
c) *$|\mu_\alpha|(\cup G_\gamma) < \varepsilon$ where for $\alpha \in [1, \omega^k]^{d(s)}$ the union is over all $\gamma \in (\alpha^-, \alpha)$,*
d) *$|\mu_\alpha(G_\alpha) - a| < \varepsilon \quad \forall \alpha \leqq \omega^k$,*
e) *$|\mu_\alpha|(G_{\omega^k} - G_\alpha) < \varepsilon \quad \forall \alpha \leqq \omega^k$.*

Then $Y = [1_{G_\alpha}]_{\alpha \leqq \omega^k}$ is a subspace of $C(K)$ which is isometric to $C(\omega^k)$ and for all $y \in Y$, $\sup\{|\langle \mu_\alpha, y \rangle| : \alpha \leqq \omega^k\} \geqq (|a| - 4\varepsilon)\|y\|$.

Consequently if $|a| > 4\varepsilon$, Y is normed by $\{\mu_\alpha\}_{\alpha \leqq \omega^k}$.

PROOF. Define an operator $T: Y \to C(\omega^k)$ by $T1_{G_\alpha} = 1_{(\alpha^-, \alpha]}$ for $\alpha \in [1, \omega^k]^{d(s)}$ $s = 0, 1, \cdots, k$, and extend linearly. It is easy to verify using a) and b) that T is an isometry onto $C(\omega^k)$.

Now suppose $y = \sum_{i=0}^{n} c_i 1_{G_{\alpha_i}}$ and that $\|y\| = |y(t)| = \|\sum_{i \in J} c_i 1_{G_{\alpha_i}}\| = |\sum_{i \in J} c_i|$ where $J = \{i \mid t \in G_{\alpha_i}\}$. There is a unique index $l \in J$ such that $G_{\alpha_l} \subset G_{\alpha_i}$ for all $i \in J$. Then by e) and the triangle inequality we have

$$|\langle y, \mu_{\alpha_l} \rangle| \geqq |\langle y \cdot 1_{G_{\alpha_l}}, \mu_{\alpha_l} \rangle| - |\langle y \cdot 1_{G_{\alpha_l}^c}, \mu_{\alpha_l} \rangle|$$

$$\geqq |y \cdot 1_{G_{\alpha_l} - \cup G_\gamma}, \mu_{\alpha_l} \rangle| - |\langle y \cdot 1_{\cup G_\gamma}, \mu_{\alpha_l} \rangle| - \varepsilon \|y\|$$

where the union is over all γ such that $G_\gamma \subsetneqq G_{\alpha_l}$;

$$|\langle y \cdot 1_{\cup G_\gamma}, \mu_{\alpha_l} \rangle| \leqq \varepsilon \|y\|$$

by c) and

$$|\langle y\cdot 1_{(G_{\alpha_l}-\cup G_\gamma)},\mu_{\alpha l}\rangle|\geqq\left|\sum_{i\in J}c_i\right|\,|\mu_{\alpha_l}(G_{\alpha_l}-\cup G_\gamma)|\geqq\|y\|(|a|-2\varepsilon)$$

by d) and c). Thus

$$|\langle y,\mu_{\alpha_l}\rangle|\geqq\|y\|(|a|-4\varepsilon).$$

Our last lemma is a major tool in our construction. Recall that when we write $A=\{a_\alpha\}_{\alpha\leqq\omega^\gamma}$ the correspondence $a_\alpha\leftrightarrow\alpha$ is a homeomorphism.

LEMMA 1.5. *Let $(A_n)_{n=1}^\infty$ be a sequence of disjoint w^*-closed subsets of $B_{C(K)^*}$, K a compact Hausdorff space, and let μ be a probability measure on K such that $\bigcup_{n=1}^\infty A_n\subset L_1(\mu)$. Further assume that $A_n=\{\mu_{n,\alpha}\}_{\alpha\leqq\omega^{\beta(n)}}$ for some ordinal $\beta(n)$ and that there are constants $a>\xi\geqq 0$ such that*

$$\lambda\left(\bigcup_{n=1}^\infty A_n,\mu\right)=a,$$

$$\lambda\left((\mu_{n,\omega^{\beta(n)}})_{n=1}^\infty,\mu\right)\geqq a-\xi.$$

Then for every $\delta>0$ there exist an infinite subset M of N, disjoint open subsets $(G_n)_{n\in M}$ of K, and subsets $\{\nu_{n,\alpha}\}_{\alpha\leqq\omega^{\beta(n)}}\subset\{\mu_{n,\alpha}\}_{\alpha\leqq\omega^{\beta(n)}}$ for each $n\in M$ such that $\lambda(\bigcup_{n\in M}\{\nu_{n,\alpha|G_n^c}:\alpha\leqq\omega^{\beta(n)}\},\mu)\leqq\xi+\delta$. Moreover if K is 0-dimensional, the sets $(G_n)_{n\in M}$ may be chosen to be clopen.

PROOF. By Lemma 1.3 there exist an infinite subset L of $\mathbf{N}$ and disjoint open subsets $(G_n)_{n\in L}$ of K such that $|\mu_{n,\omega^{\beta(n)}}|(G_n)>a-\xi-\delta/4$ and $|\mu_{n,\omega^{\beta(n)}}|(\bigcup_{l\neq n}G_l)<\delta/4$ for all $n\in L$. For each $n\in L$, A_n is homeomorphic to $[1,\omega^{\beta(n)}]$. Hence any w^* closed neighborhood of $\mu_{n,\omega^{\beta(n)}}$ in A_n is again homeomorphic to $[1,\omega^{\beta(n)}]$. For each $n\in L$ choose a function $f_n\in B_{C(K)}$ such that $f_{n|G_n^c}=0$ and

$$|\langle\mu_{n,\omega^{\beta(n)}},f_n\rangle-|\mu_{n,\omega^{\beta(n)}}|(G_n)|<\delta/4.$$

Let $B_n=\{\mu_{n,\alpha}:|\langle\mu_{n,\alpha},f_n\rangle-\langle\mu_{n,\omega^{\beta(n)}},f_n\rangle|\leqq\delta/4\}$.

We claim that there is an l_0 such that if $M=L\cap\{l\geqq l_0\}$

$$\lambda\left(\bigcup_{m\in M}\{\mu_{n,\alpha|G_n^c}:\mu_{n,\alpha}\in B_n\},\mu\right)\leqq\xi+\frac{3\delta}{4}.$$

If not, by Lemma 1.1 we can find a sequence of measures $(\mu_{n,\alpha_n})_{n\in L}$ with $\mu_{n,\alpha_n}\in B_n$ such that

$$\lambda((\mu_{n,\alpha_n|G_n^c})_{n\in L}, \mu) > \xi + \frac{3\delta}{4}.$$

Moreover, since $\mu_{n,\alpha_n} \in B_n$, we have that

$$\overline{\lim_L} |\mu_{n,\alpha_n}|(G_n) > \overline{\lim_L} |\mu_{n,\omega^{\beta(n)}}|(G_n) - \delta/4 \geqq a - \xi - \delta/2.$$

Thus since $\mu(G_n) \to 0$,

$$\lambda((\mu_{n,\alpha_n})_{n\in L}, \mu) > (a - \xi - \delta/2) + (\xi + 3\delta/4) = a + \delta/4,$$

an impossibility. Consequently with M as above and $\{\nu_{n,\alpha}\}_{\alpha \leqq \omega^{\beta(n)}} = B_n$ for each $n \in M$, we have satisfied the requirements of the lemma.

If K is 0-dimensional by Remark 1 following Lemma 1.3 we can choose the sets $(G_n)_{n\in M}$ to be clopen.

2. Proof of the theorem on $C(\omega^\omega)$

Before proceeding to the proof of Theorem 0.2 we will make a few observations and outline the argument.

First note that in view of Rosenthal's result [10] we may assume that $T^*B_{X^*}$ is separable and consequently that there is a probability measure μ such that for all $\nu \in T^*B_{X^*}$, $\nu \ll \mu$. Second we will assume that $\|T^*\| \leqq 1$. Indeed, there is no loss of generality since $\eta(\varepsilon, B_{C(K)}, T^*B_{X^*}) \geqq \omega$ if and only if $\eta(\varepsilon \|T^*\|^{-1}, B_{C(K)}, \|T^*\|^{-1}T^*B_{X^*}) \geqq \omega$. Finally we will assume that $K = \Delta$, the Cantor set. This assumption will allow us to use clopen sets and thus avoid some minor technical difficulties in constructing the subspace Y. We describe the modifications necessary to handle arbitrary compact metric spaces K in the remark at the end of this section.

We divide the proof into two parts. In the first we prove a finite index version of Theorem 0.2 and in the second we show how to combine copies of $C(\omega^n)$ to get the required copy of $C_0(\omega^\omega)$. In each case we use the condition on the index to produce a subset K of $T^*B_{X^*}$, homeomorphic to $[1, \omega^n]$ in the first case and $[1, \omega^\omega]$ in the second, so that each sequence in K is not uniformly absolutely continuous with respect to μ. Then we construct a subspace Y of $C(\Delta)$ such that Y is normed by K. In the finite index case we will actually find a subset K' of K with K' homeomorphic to $[1, \omega^{k(n)}]$ so that the evaluation map from Y into $C(K')$ is an isomorphism onto $C(K')$ and $k(n) \to \infty$ as $n \to \infty$. In the infinite index case we will again get a subset K' of K but K' will be homeomorphic $[1, \omega^\omega]$ and the evaluation map from Y into $C(K')$ will be an isomorphism onto $C_0(K')$.

PROPOSITION 2.1. *For every $\varepsilon > 0$ and integer k there is an $n = n(\varepsilon, k)$ such that if $T: C(\Delta) \to X$, where X is an arbitrary Banach space, $\|T\| \leqq 1$, and $\eta(\varepsilon, B_{C(\Delta)}, T^* B_{X^*}) \geqq n$, then there is a subspace Y_k of $C(\Delta)$ such that Y_k is isometric to $C(\omega^k)$, $T_{|Y_k}$ is an isomorphism and $\|(T_{|Y_k})^{-1}\|$ depends only on ε.*

PROOF. We will see how to choose $n(\varepsilon,k)$ later. We have that $P_n(\varepsilon) = P_n(\varepsilon, B_{C(\Delta)}, T^* B_{X^*}) \neq \emptyset$. Let $\nu_{\omega^n} \in P_n(\varepsilon)$ and choose a sequence $(\nu_{\omega^{n-1}l})_{l=1}^{\infty} \subset P_{n-1}(\varepsilon)$ such that $\nu_{\omega^{n-1}l} \xrightarrow{w^*} \nu_{\omega^n}$ and there is a sequence $(f_{\omega^{n-1}l})_{l=1}^{\infty} \subset B_{C(\Delta)}$, $f_{\omega^{n-1}l} \xrightarrow{w} 0$, with $\overline{\lim}_{l\to\infty} \langle \nu_{\omega^{n-1}l}, f_{\omega^{n-1}l} \rangle \geqq \varepsilon$. For each l choose a sequence $(\nu_{\omega^{n-1}(l-1)+\omega^{n-2}m})_{m=1}^{\infty} \subset P_{n-2}(\varepsilon)$ such that $\nu_{\omega^{n-1}(l-1)+\omega^{n-2}m} \xrightarrow{w^*} \nu_{\omega^{n-1}l}$ and there is a sequence $(f_{\omega^{n-1}(l-1)+\omega^{n-2}m})_{m=1}^{\infty} \subset B_{C(\Delta)}$, $f_{\omega^{n-1}(l-1)+\omega^{n-2}m} \xrightarrow{w} 0$, with $\overline{\lim}_{m\to\infty} \langle \nu_{\omega^{n-1}(l-1)+\omega^{n-2}m}, f_{\omega^{n-1}(l-1)+\omega^{n-2}m} \rangle \geqq \varepsilon$. Using the metrizability of the w^* topology we may assume that for any sequence $(\nu_{\omega^{n-1}l+\omega^{n-2}m(l)})_{l=1}^{\infty}$, $\nu_{\omega^{n-1}l+\omega^{n-2}m(l)} \xrightarrow{w^*} \nu_{\omega^n}$.

By repeating this process n times we get a subset $\{\nu_\alpha\}_{\alpha \leqq \omega^n}$ of $T^* B_{X^*}$ such that if $\alpha \in [1, \omega^n]^{(r)}$, $\nu_\alpha \in P_r(\varepsilon)$, and, for any integers $l(1), l(2), \cdots, l(r)$, and $\beta = \omega^{n-1} l(1) + \omega^{n-2} l(2) + \cdots + \omega^{n-r} l(r)$ there is a sequence $(f_{\beta+\omega^{n-r-1}l})_{l=1}^{\infty} \subset B_{C(\Delta)}$, $f_{\beta+\omega^{n-r-1}l} \xrightarrow{w} 0$ such that $\overline{\lim} \langle \nu_{\beta+\omega^{n-r-1}l}, f_{\beta+\omega^{n-r-1}l} \rangle \geqq \varepsilon$.

Our set $(\nu_\alpha)_{\alpha \leqq \omega^n}$ is not entirely satisfactory because we do not know that for any sequence $(\nu_{\alpha_r})_{r=1}^{\infty}$, there is a sequence of functions $(g_r)_{r=1}^{\infty}$, such that $g_r \xrightarrow{w} 0$ and $\overline{\lim}_r \langle v_{\alpha_r}, g_r \rangle \geqq \varepsilon$.

Observe that if $(g_r)_{r=1}^{\infty} \subset C(\Delta)$ and $g_r \xrightarrow{w} 0$ then $g_r \to 0$ in measure μ. For us it is enough to have our set $\{\nu_\alpha\}_{\alpha \leqq \omega^n}$ satisfy the following property:

(1) If $(\nu_{\alpha_r})_{r=1}^{\infty}$ is a sequence of distinct elements such that $\nu_{\alpha_r} \xrightarrow{w^*} \nu_\alpha$, then there is a sequence $(g_r)_{r=1}^{\infty} \subset B_{C(\Delta)}$ such that $g_r \to 0$ in measure μ and $\overline{\lim}_{r\to\infty} \langle \nu_{\alpha_r}, g_r \rangle \geqq \varepsilon$.

To accomplish this we note that convergence in measure μ is a metric convergence. Let $d(\,,)$ be a metric on $C(\Delta)$ for the topology of convergence in measure μ (we may assume without loss of generality that $\mu(G) > 0$ for all open sets G of Δ). Now by passing to a subsequence of $(\nu_{\omega^{n-1}l})_{l=1}^{\infty}$ we may assume that $d(f_{\omega^{n-1}l}, 0) < 1/2^l$ and $\langle \nu_{\omega^{n-1}l}, f_{\omega^{n-1}l} \rangle \geqq \varepsilon - 1/2^l$ for $l = 1, 2, \cdots$. Similarly for each l we can assume by passing to a subsequence of $(\nu_{\omega^{n-1}l+\omega^{n-2}m})_{m=1}^{\infty}$ that $d(f_{\omega^{n-1}l+\omega^{n-2}m}, 0) < 1/2^{l+m}$ and

$$\langle \nu_{\omega^{n-1}l+\omega^{n-2}m}, f_{\omega^{n-1}l+\omega^{n-2}m} \rangle > \varepsilon - \frac{1}{2^{l+m}}.$$

Thus for every sequence $(\nu_{\omega^{n-1}l+\omega^{n-2}m(l)})_{l=1}^{\infty}$,

$$\lim_{l\to\infty} \langle \nu_{\omega^{n-1}l+\omega^{n-2}m(l)}, f_{\omega^{n-1}l+\omega^{n-2}m(l)}\rangle \geqq \varepsilon$$

and $f_{\omega^{n-1}l+\omega^{n-2}m(l)} \to 0$ in measure μ.

Consequently by refining $\{\nu_\alpha\}_{\alpha\leqq\omega^n}$ in the manner described above we can assume that (1) is satisfied. This in turn tells us that for any sequence of distinct elements $(\nu_{\alpha_r})_{r=1}^{\infty}$, $\lambda((\nu_{\alpha_r})_{r=1}^{\infty},\mu)\geqq\varepsilon$. Indeed, if we have a sequence $(\nu_{\alpha_r})_{r=1}^{\infty}$ and a sequence $(g_r)_{r=1}^{\infty}\subset B_{C(\Delta)}$ such that $g_r \xrightarrow{\mu} 0$ and $\overline{\lim}_{r\to\infty}\langle \nu_{\alpha_r}, g_r\rangle \geqq \varepsilon$, then for every $\rho>0$ we can find a subset K_1 of Δ such that $g_r\to 0$ uniformly on K_1 and $\mu(\Delta-K_1)<\rho$. Then $\langle \nu_{\alpha_r}, g_r\rangle \leqq \|g_{r|K_1}\|\,\|\nu_{\alpha_r}\| + |\nu_{\alpha_r}|(\Delta-K_1)$ and thus we can find an r such that $|\nu_{\alpha_r}|(\Delta-K_1)\geqq\varepsilon-\rho$. Hence $\lambda((\nu_{\alpha_r})_{r=1}^{\infty},\mu)\geqq\varepsilon$.

Let $M=\{\phi_\alpha\}_{\alpha\leqq\omega^r}$ be a subset of $B_{C(\Delta)^*}$. Then for each

$$\beta\in[1,\omega^r]^{d(s)},\qquad s=1,2,\cdots,r \tag{2}$$

let

$$M(s,\beta)=\{\phi_\alpha \mid \beta^- <\alpha<\beta\}$$

(recall that β^- was defined prior to Lemma 1.4). Note that for each s

$$\cup\{M(s,\beta):\beta\in[1,\omega^r]^{d(s)}\}=M-M^{(s)}$$

and if we fix $\beta\in[1,\omega^r]^{d(s)}$

$$M(s,\beta)=\cup\{M(s-1,\gamma):\gamma\in(\beta^-,\beta]^{d(s-1)}\}\cup\{\phi_\gamma \mid \gamma\in(\beta^-,\beta]^{d(s-1)}\}.$$

The main difficulty in our proof is to construct a subset M of $\{\nu_\alpha\}_{\alpha\leqq\omega^n}$, $M=\{\phi_\alpha\}_{\alpha\leqq\omega^{k+1}}$ (i.e., $r=k+1$ above) and a number $a\geqq 3\varepsilon/4$ such that

$$|\lambda((M(s,\beta))^{d(t)},\mu)-a|<\zeta \tag{3}$$

for all s,t,β such that $0\leqq t<s\leqq k+1$ and $\beta\in[1,\omega^{k+1}]^{d(s)}$, where $\zeta=\varepsilon\cdot 2^{-k-1}/240$. Let us assume that we have found M and a. We will show that there are clopen sets $\{G_\alpha\}_{\alpha\leqq\omega^k}$ satisfying the hypotheses of Lemma 1.4 with $\varepsilon/16$ replacing ε.

Before we begin the construction of Y_k and the set $\{G_\alpha\}_{\alpha\leqq\omega^k}$ let us make a few observations. First although M is homeomorphic to $[1,\omega^{k+1}]$ the conditions on M will only allow us to build $C_0(\omega^{k+1})$. Consequently we will pass to a subspace Y_k isometric to $C(\omega^k)$. Second let us examine the relationship between the conditions of Lemma 1.4 and the conditions we have imposed on M. To simplify matters we will assume $\mu_\alpha\geqq 0$ for all $\alpha\leqq\omega^k$ in Lemma 1.4.

Conditions d) and e) of Lemma 1.4 imply that $|\mu_\alpha(G_{\omega^k})-a|<2\varepsilon$ for each $\alpha\leqq\omega^k$ and hence that for any sequence $(\mu_{\alpha_n})_{n=1}^\infty$, $\lambda((\mu_{\alpha_n|G_{\omega^k}})_{n=1}^\infty,\mu)<a+2\varepsilon$. Also we have that for any s tuple of integers $(l(1),l(2),\cdots,l(s))$ and $\beta=\omega^{k-1}l(1)+\cdots+\omega^{k-s}l(s)$, $\mu_{\beta+\omega^{k-s-1}r}(G_{\beta+\omega^{k-s-1}r})>a-\varepsilon$ for $r=1,2,\cdots$ by d) and the sets $(G_{\beta+\omega^{k-s-1}r})_{r=1}^\infty$ are disjoint by b). Thus $\lambda((\mu_{\beta+\omega^{k-s-1}r})_{r=1}^\infty,\mu)\geqq a-\varepsilon$. Now consider $(\phi_{\beta+\omega^{k-s-1}r})_{r=1}^\infty=[M(k,\omega^k)]^{d(k-1)}$. Our assumption on M yields

$$|\lambda((\phi_{\beta+\omega^{k-s-1}r})_{r=1}^\infty,\mu)-a|<\zeta.$$

Also since

$$M-\{\phi_{\omega^{k+1}}\}=M(k+1,\omega^{k+1})=\bigcup_{t=0}^{k}[M(k+1,\omega^k)]^{d(t)}$$

we have that

$$\lambda(M,\mu)=\lambda\left(\bigcup_{t=0}^{k}[M(k+1,\omega^{k+1})]^{d(t)},\mu\right)$$

$$=\max_{0\leqq t\leqq k}\lambda([M(k+1,\omega^{k+1})]^{d(t)},\mu)\leqq a+\zeta.$$

Consequently for any sequence $(\phi_{\alpha_n})_{n=1}^\infty$, $\lambda((\phi_{\alpha_n})_{n=1}^\infty,\mu)\leqq a+\zeta$.

Our more restrictive condition,

$$|\lambda([M(s,\beta)]^{d(t)},\mu)-a|<\zeta,$$

is made necessary by our "level by level" construction of the sets $\{G_\alpha\}_{\alpha\leqq\omega^k}$. By this we mean that we will first find G_{ω^k} and μ_{ω^k}, then we find a sequence of disjoint clopen subsets $(G_{\omega^{k-1}l})_{l=1}^\infty$ of G_{ω^k} and measures $(\mu_{\omega^{k-1}l(1)})_{l(1)=1}^\infty$. For each $l(1)$ we find a sequence of disjoint clopen subsets $(G_{\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)})_{l(2)=1}^\infty$ of $G_{\omega^{k-1}l(1)}$ and measures $(\mu_{\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)})_{l(2)=1}^\infty$ and so forth. In this way the functions $\{1_{G_\alpha}\}_{\alpha\leqq\omega^k}$ behave exactly as the family of functions $\mathcal{F}$ that we defined prior to Lemma 1.4 (with $n=k$ in this case) and the measures $\{\mu_\alpha\}_{\alpha\leqq\omega^k}$ behave like the point masses $\{\delta_\alpha\}_{\alpha\leqq\omega^k}$. We will now construct these measures and sets.

Let $A_l=\{\phi_\alpha\,|\,\omega^k(l-1)<\alpha\leqq\omega^k l\}$, $l=1,2,\cdots$ and $\xi=2\zeta$, then

$$\lambda\left(\bigcup_{l=1}^{\infty}A_l,\mu\right)=\lambda\left(\bigcup_{t=0}^{k}[M(k+1,\omega^{k+1})]^{d(t)},\mu\right)$$

$$\leqq a+\zeta\quad\text{and}\quad\lambda((\phi_{\omega^k l})_{l=1}^\infty,\mu)$$

$$=\lambda([M(k+1,\omega^{k+1})]^{d(k)},\mu)>a-\zeta.$$

Thus by Lemma 1.5 with $\delta=\zeta$ we get an infinite subset $L\subset N$, disjoint clopen sets $(G_{\omega^k l})_{l\in L}$ and subsets $\{\psi_\alpha\,|\,\omega^k(l-1)<\alpha\leqq\omega^k l\}$ of A_l, $l\in L$, such that

(4) $$\lambda\left(\bigcup_{l\in L}\{\psi_{\alpha|G^c_{\omega^k l}}:\omega^k(l-1)<\alpha\leqq\omega^k l\},\mu\right)<\zeta.$$

If we examine the proof of Lemma 1.5 we see that the subset $\{\psi_\alpha:\omega^k(l-1)<\alpha\leqq\omega^k l\}$ is in fact a neighborhood of $\phi_{\omega^k l}$. Thus we may assume that it is of the form $\{\phi_\alpha:\omega^k(l-1)+\omega^{k-1}r<\alpha\leqq\omega^k l\}$ for some integer r. If $\beta\in[\omega^k(l-1)+\omega^{k-1}r,\omega^k l]^{d(s)}$ for $s<k$, $M(s,\beta)\subset\{\phi_\alpha:\omega^{k-1}(l-1)+\omega^{k-1}r<\alpha\leqq\omega^k l\}$. Consequently this neighborhood has the same properties as the set A_l. So we may assume without loss of generality that the subset $\{\psi_\alpha:\omega^k(l-1)<\alpha<\omega^k l\}$ is in fact A_l itself.

By passing to an infinite subset L' of L we can assume that

(5) $$\phi_{\alpha|G^c_{\omega^k j}}|\left(\bigcup_{l\in L'}G_{\omega^k l}\right)<4\zeta$$

$\forall\alpha\in(\omega^k(j-1),\omega^k j]$ and $j\in L'$, and consequently we can assume that this is the case for L itself. (Recall the observation we made following the definition of $\lambda(,)$.) Also since $\lambda(\bigcup_{l=1}^\infty A_l,\mu)\leqq a+\zeta$ we can assume for all $\alpha\in(\omega^k(l-1),\omega^k l]$ that

(6) $$|\phi_\alpha|(G_{\omega^k l})<a+2\zeta\quad\text{for all}\quad l\in L.$$

Fix $l\in L$ and find disjoint clopen subsets of $G_{\omega^k l}$, say, $G^+_{\omega^k l}$ and $G^-_{\omega^k l}$ (which are almost the positive and negative sets for $\phi_{\omega^k l}$), such that

(7) $$|\phi_{\omega^k l}|G^+_{\omega^k l}-\phi_{\omega^k l}(G^+_{\omega^k l})<\zeta\quad\text{and}$$
$$\phi_{\omega^k l}(G^-_{\omega^k l})+|\phi_{\omega^k l}|G^-_{\omega^k l}<\zeta.$$

This splitting of $G_{\omega^k l}$ fixes the sign of the functions we will take to span $C(\omega^k)$. For the rest of the argument we will assume that $G^+_{\omega^k l}=G_{\omega^k l}$ rather than carry out the construction on both $G^+_{\omega^k l}$ and $G^-_{\omega^k l}$. (Also, we will not use any of the sets $G_{\omega^k s}$, $s\neq l$, $s\in L$, in the remainder of the argument.)

Our next step is to find a sequence of disjoint clopen subsets $(G_{\omega^k(l-1)+\omega^k l(1)})_{l(1)=1}^\infty$ of $G_{\omega^k l}$ which will form the second level of sets in our construction.

Choose m such that

(8) $$|\langle\phi_{\omega^k l}-\phi_{\omega^k(l-1)+\omega^{k-1}l(1)},1_{G_{\omega^k l}}\rangle|<\zeta$$

for all $l(1)\geqq m$. Note that

$$(\phi_{\omega^k(l-1)+\omega^k l(1)})_{l(1)=1}^\infty=[M(k,\omega^k l)]^{d(k-1)},$$

so that by (3) and (4)

$$\lambda\left(\left(\phi_{\omega^k(l-1)+\omega^{k-1}l(1)|G_{\omega^k l}}\right)_{l(1)\geqq m}, \mu\right) > a - 4\zeta.$$

Thus using Lemma 1.3 we can find an infinite set $L(1) \subset \mathbf{N}$ and disjoint clopen subsets of $G_{\omega^k l}$, say, $(G_{\omega^k(l-1)+\omega^{k-1}l(1)})_{l(1)\in L(1)}$ such that

$$(9) \qquad \left|\phi_{\omega^k(l-1)+\omega^{k-1}l(1)}\right|\left(G_{\omega^k(l-1)+\omega^{k-1}l(1)}\right) > a - 5\zeta$$

and

$$\left|\phi_{\omega^k(l-1)+\omega^{k-1}l(1)}\right|\left(\bigcup G_{\omega^k(l-1)+\omega^{k-1}r}\right) < \zeta$$

where the union is over all $r \in L(1) - \{l(1)\}$. Hence by (6) and (9)

$$(10) \qquad \left|\phi_{\omega^k(l-1)+\omega^{k-1}l(1)}\right|\left(G_{\omega^k l} - G_{\omega^k(l-1)+\omega^{k-1}l(1)}\right) < 7\zeta$$

and

$$(11) \qquad \begin{aligned} &\phi_{\omega^k(l-1)+\omega^{k-1}l(1)}\left(G_{\omega^k(l-1)+\omega^{k-1}l(1)}\right) \\ &\geqq \phi_{\omega^k(l-1)+\omega^{k-1}l(1)}(G_{\omega^k l}) - \left|\phi_{\omega^k(l-1)+\omega^{k-1}l(1)}\right|\left(G_{\omega^k l} - G_{\omega^k(l-1)+\omega^{k-1}l(1)}\right) \\ &\geqq \phi_{\omega^k l}(G_{\omega^k l}) - \zeta - 7\zeta > a - 15\zeta, \end{aligned}$$

by (8), (10), and (7) (we can assume $|\phi_{\omega^k l}|(G_{\omega^k l}) > a - 5\zeta$ by (3) and (4)). Also by passing to an infinite subset of $L(1)$ we may assume that

$$|\phi_{\omega^k l}| \bigcup_{l(1)\in L(1)} G_{\omega^k(l-1)+\omega^{k-1}l(1)} < \zeta.$$

Our second level of sets and measures is now complete.

For each $l(1) \in L(1)$ choose $m = m(l(1))$ such that

$$(12) \qquad \left|\left\langle \phi_{\omega^k(l-1)+\omega^{k-1}l(1)} - \phi_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)}, 1_{G_{\omega^k(l-1)+\omega^{k-1}l(1)}}\right\rangle\right| < \zeta$$

for all $l(2) \geqq m$. As before

$$(13) \qquad \lambda\left(\left(\phi_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)}\right)_{l(2)\geqq m}, \mu\right) > a - \zeta$$

since

$$\left(\phi_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)}\right)_{l(2)=1}^{\infty} = \left[M(k-1, \omega^k(l-1) + \omega^{k-1}l(1))\right]^{d(k-2)}.$$

So

$$\begin{aligned} &\lambda\left(\left(\phi_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)|G_{\omega^k(l-1)+\omega^{k-1}l(1)}}\right)_{l(2)\geqq m}, \mu\right) \\ &\geqq a - \zeta - \lambda\left(\left(\phi_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)|G^c_{\omega^k(l-1)+\omega^{k-1}l(1)}}\right)_{l(2)\geqq m}, \mu\right) \\ &\geqq a - \zeta - 15\zeta - 6\zeta = a - 22\zeta \end{aligned}$$

because

$$\phi_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)}(G_{\omega^k(l-1)+\omega^{k-1}l(1)}) > a - 15\zeta - \zeta,$$

by (11) and (12),

$$|\phi_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)}|(G_{\omega^k l}) < a + 2\zeta,$$

by (6), and by (4)

$$\lambda((\phi_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)|G^c_{\omega^k l}})_{l(2)=1}^{\infty}, \mu) < 3\zeta.$$

Again by using Lemma 1.3 we can find an infinite subset $L(l(1))$ of $\mathbf{N}$ and clopen subsets $(G_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)})_{l(2)\in L(l(1))}$ of $G_{\omega^k(l-1)+\omega^{k-1}l(1)}$ such that

$$(14) \qquad |\phi_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)}|(G_{\omega^k(l-1)+\omega^k(l(1)-1)+\omega^{k-2}l(2)}) > a - 23\zeta$$

and

$$(15) \qquad |\phi_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)}|(\cup G_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}r}) < \zeta$$

where the union is over all $r \in L(l(1)) - \{l(2)\}$.

We may also assume (by discarding a finite subset of $L(l(1))$) that

$$|\phi_{\omega^k(l-1)+\omega^{k-1}l(1)}| \bigcup_{l(2)\in L(l(1))} G_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)} < \zeta.$$

Hence by (14)

$$|\phi_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)}|(G_{\omega^k l} - G_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)}) < 25\zeta$$

since

$$|\phi_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)}|(G_{\omega^k l}) < a + 2\zeta,$$

by (6), and

$$\begin{aligned}
&\phi_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)}(G_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)}) \\
&\quad \geqq \phi_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)}(G_{\omega^k(l-1)+\omega^{k-1}l(1)}) \\
&\qquad - |\phi_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)}|(G_{\omega^k(l-1)+\omega^{k-1}l(1)} - G_{\omega^k(l-1)+\omega^{k-1}(l(1)-1)+\omega^{k-2}l(2)}) \\
&\quad \geqq \phi_{\omega^k(l-1)+\omega^k l(1)}(G_{\omega^k(l-1)+\omega^{k-1}l(1)}) - \zeta - 23\zeta - 2\zeta \\
&\quad \geqq a - 15\zeta - 26\zeta = a - 41\zeta.
\end{aligned}$$

Here we have used (12), (14), (6), and (11).

Continuing in this fashion we obtain a subset $\{\mu_\alpha\}_{\alpha\leqq\omega^k}$ of $\{\phi_\alpha : \omega^k(l-1) < \alpha \leqq \omega^k l\}$ and clopen subsets $(H_\alpha)_{\alpha\leqq\omega^k}$ of $G_{\omega^k l}$ such that for any s-tuple of integers $l(1), l(2), \cdots, l(s)$, $s \leqq k$:

a′)
$$H_{\omega^{k-1}l(1)+\omega^{k-2}l(2)+\cdots+\omega^{k-s}l(s)}$$
$$\subsetneqq H_{\omega^{k-r}l(1)+\omega^{k-2}l(2)+\cdots+\omega^{k-r}l(r)}$$
$$\subsetneqq H_{\omega^k} = G_{\omega^k l} \qquad 1 \leqq r < s;$$

b′)
$$H_{\omega^{k-1}l(1)+\omega^{k-2}l(2)+\cdots+\omega^{k-s}l(s)}$$
$$\cap H_{\omega^{k-1}m(1)+\omega^{k-2}m(2)+\cdots+\omega^{k-s}m(s)} = \emptyset$$

if $(m(1), m(2), \cdots, m(s))$ is different from $(l(1), l(2), \cdots, l(s))$;

c′)
$$|\mu_{\omega^{k-1}l(1)+\omega^{k-2}l(2),\cdots,\omega^{k-s}l(s)}| \bigcup_{r=1}^{\infty} H_{\omega^{k-1}l(1)+\omega^{k-2}l(2)+\cdots+\omega^{k-s}l(s)+\omega^{k-s-1}r}$$
$$< \zeta = \varepsilon 2^{-k-1}/240;$$

d′)
$$|\mu_{\omega^{k-1}l(1)+\omega^{k-2}l(2)+\cdots+\omega^{k-s}l(s)}(H_{\omega^{k-1}l(1)+\omega^{k-2}l(2)+\cdots+\omega^{k-s}l(s)}) - a|$$
$$< (2^s-1)15\zeta + 10\zeta \leqq (2^{s+1}-1)\varepsilon 2^{-k-1}/16;$$

e′)
$$|\mu_{\omega^{k-1}l(1)+\omega^{k-2}l(2)+\cdots+\omega^{k-s}l(s)}|(H_{\omega^k} - H_{\omega^{k-1}l(1)+\omega^{k-2}l(2)+\cdots+\omega^{k-s}l(s)})$$
$$< (2^s-1)15\zeta + 10\zeta = (2^s-1)\varepsilon 2^{-k-1}/16 + 10\varepsilon 2^{-k-1}/240.$$

Hence the hypotheses of Lemma 1.4 are satisfied for a and $\varepsilon/16$.

If we had not assumed $G^+_{\omega^k l} = G_{\omega^k l}$ then for the measures

$$\rho_\alpha = \phi_\alpha \cdot (1_{G^+_{\omega^k l}} - 1_{G^-_{\omega^k l}}), \qquad \omega^k(l-1) < \alpha \leqq \omega^k l,$$

we could complete the argument as above. Clearly the space

$$Z = \{y \cdot (1_{G^+_{\omega^k l}} - 1_{G^-_{\omega^k l}}) \mid y \in Y\}$$

would be isometric to $C(\omega^k)$ and normed by $\{\phi_\alpha : \omega^k(l-1) < \alpha \leqq \omega^k l\}$.

REMARK. From Lemma 1.4 we get the estimate $\|T^{-1}_{1Y}\| \leqq 1/(a-\varepsilon/4) \leqq 2/\varepsilon$. It is easy to see that by sharpening our estimates that for any $\varepsilon' < \varepsilon$ we can find a subspace Y such that $\|T^{-1}_{1Y}\| \leqq 1/\varepsilon'$.

It remains to show that we can find the set M satisfying (3).

Let $F = \{\nu_\alpha\}_{\alpha\leqq\omega^n}$ and for each i, j, β, $0 \leqq i < j \leqq n$, $\beta \in [1, \omega^n]^{d(j)}$ consider the numbers

$$\lambda(F(j,\beta)^{d(i)}, \mu)$$

where $F(j,\beta)$ is defined analogously to $M(j,\beta)$, i.e., as in (2), $F(j,\beta)=\{\nu_\alpha \mid \beta^- < \alpha < \beta\}$. Our construction of M is in three steps:

(i) Find numbers a_{ji} such that

$$|\lambda(F(j,\beta)^{d(i)},\mu)-a_{ji}|<\zeta/8 \quad \forall\beta\in[1,\omega^n]^{d(j)},$$

$0\leqq i<j\leqq n$ by passing to a subset of F of the same homeomorphic type.

(ii) Use Ramsey's Theorem on (a_{ji}) to find indices $i(0)<i(1)<\cdots<i(k+1)$ with $a_{i(s)i(t)}=c$.

(iii) Find a subset M of F homeomorphic to $[1,\omega^{k+1}]$ such that

$$|\lambda(M(s,\beta)^{d(t)},\mu)-c|<\zeta.$$

Step i) We use the following lemma:

Lemma 2.2. *For every n and $\delta>0$ if $F=\{\nu_\alpha\}_{\alpha\leqq\omega^n}$, $0\leqq\|\nu\|\leqq 1$, $\forall\nu\in F$, and $c_1,c_2,\cdots,c_r$ is a δ-net in $[0,1]$ there is a subset D of F and indices $r(j,i)\in\{1,2,\cdots,r\}$ such that*

1) *D is homeomorphic to $[1,\omega^n]$,*

2) *$|\lambda(D(j,\beta)^{d(i)},\mu)-c_{r(j,i)}|<\delta$ for all i,j,β, $0\leqq i<j\leqq n$ and $\beta\in[1,\omega^n]^{d(j)}$.*

Proof. We use induction on n. If $n=1$, F is a sequence $(\nu_k)_{k=1}^\infty$ and its limit ν_ω and there is only $\lambda(F(1,\omega)^{d(0)},\mu)$ to consider. Since $\{c_1,c_2,\cdots,c_r\}$ is a δ-net in $[0,1]$ there is an integer $r(1,0)\in\{1,2,\cdots,r\}$ such that $|\lambda(F(1,\omega)^{d(0)},\mu)-c_{r(1,0)}|<\delta$. Thus $D=F$ satisfies 1) and 2) with $r(1,0)$.

Now suppose the lemma is true for $n-1$ and F is homeomorphic to $[1,\omega^n]$. Then the set $E_l=\{\nu_\alpha:\omega^{n-1}(l-1)<\alpha\leqq\omega^{n-1}l\}$ is homeomorphic to $[1,\omega^{n-1}]$, for each l. Since the lemma is valid for $n-1$, for each l we can find a set of indices $\{r(j,i,l):0\leqq i<j\leqq n-1\}$ and a subset D_l of E_l homeomorphic to $[1,\omega^{n-1}]$ such that

$$|\lambda(D_l(j,\beta)^{d(i)},\mu)-c_{r(j,i,l)}|<\delta.$$

Since there are only finitely many pairs (j,i), $0\leqq i<j\leqq n-1$ we can find an infinite subset L of N such that

$$r(j,i,l)=r(j,i,m) \quad \text{for all} \quad l,m\in L,$$

$0\leqq i<j\leqq n-1$. Let $D=\overline{\bigcup_{l\in L}D_l}$ (which is clearly homeomorphic to $[1,\omega^n]$). Then for all i,j,β such that $0\leqq i<j\leqq n-1$ and $\beta\in[1,\omega^n]^{d(j)}$

$$D(j,\beta)=D_l(j,\beta) \quad \text{for some} \quad l\in L$$

and hence

$$|\lambda(D(j,\beta)^{d(t)},\mu)-c_{r(j,i,l)}|<\delta.$$

Also for each i, $i=0,1,\cdots,n-1$ there is an index $r(n,i)\in\{1,2,\cdots,r\}$ such that

$$|\lambda(D(n,\omega^n)^{d(t)},\mu)-c_{r(n,i)}|<\delta.$$

Thus letting $r(j,i)=r(j,i,l)$, $l\in L$, $0\leqq i<j\leqq n-1$ we have proved the lemma for n.

To complete Step i) we apply Lemma 2.2 to F with $\{c_1,c_2,\cdots,c_r\}$ an $\zeta/8$-net in $[0,1]$ such that for $l=2,3,\cdots,r$, $c_l-c_{l-1}=\zeta/8$ and let $a_{ji}=c_{r(j,i)}$, $0\leqq i<j\leqq n$.

Step ii) By Ramsey's Theorem [3], if n is sufficiently large there are indices $i(0)<i(1)<\cdots<i(k+1)$ and m, $1\leqq m\leqq r$ such that

$$a_{i(s),i(t)}=c_m \quad \text{for all} \quad s,t, \quad 0\leqq t<s\leqq k+1.$$

Step iii) In this step we must find the set M so that

$$|\lambda(M(s,\beta)^{d(t)},\mu)-c\,|<\zeta.$$

This would be easy if the indices $i(0),i(1),\cdots,i(k+1)$ were consecutive. Indeed, suppose $i(k+1)=l$, $i(k)=l-1,\cdots,i(0)=l-k-1$, and consider $M=D(l,\omega^l)^{(l-k-1)}$. Then for each s,t,

$$M(s,\beta)^{d(t)}=D(l-k-1+s,\omega^{l-k-1}\beta)^{d(l-k-1+t)}$$

and consequently $|\lambda(M(s,\beta)^{d(t)},\mu)-a_{i(s),i(t)}|<\zeta$.

To handle the nonconsecutive case we use

LEMMA 2.3. *Let $A\subset B_{C(\Delta)^*}$ and let μ be a probability measure such that $A\subset L_1(\mu)$. Suppose $A=\bigcup_{l=1}^{\infty}A_l\cup\{\nu\}$ and there exists a $\delta>0$ such that $|\lambda(A_l)-\lambda(A_m)|<\delta$ for all l,m. Further assume that if $(a_l)_{l=1}^{\infty}\subset A$ such that $a_l\in A_l$ then $\lim_{l\to\infty}a_l=\nu$. Then there is a sequence $(\nu_n)_{n=1}^{\infty}\subset A$ such that $\nu_n\to\nu$ and*

$$|\lambda((\nu_n)_{n=1}^{\infty},\mu)-\lambda(A,\mu)|<3\delta.$$

PROOF. We divide the argument into two cases:

(I) $$\overline{\lim_{l\to\infty}}\,\lambda(A_l,\mu)>\lambda(A,\mu)-2\delta.$$

For each l, $\lambda(A_l,\mu)>\lambda(A,\mu)-3\delta$. By Lemma 1.1 for each l there are a sequence $(\nu_{l,k})_{k=1}^{\infty}$ and disjoint closed sets $(A_{l,k})_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty}|\nu_{l,k}|(A_{l,k})=\lambda(A_l,\mu).$$

For each l choose $k(l)$ such that

$$|\nu_{l,k(l)}|(A_{l,k(l)}) > \lambda(A,\mu) - 3\delta$$

and

$$\mu(A_{l,k(l)}) < 1/l$$

and let $\nu_l = \nu_{l,k(l)}$. Clearly $\lim_{l\to\infty} \nu_l = \nu$ and

$$\lambda(A,\mu) \geqq \lambda((\nu_l)_{l=1}^{\infty},\mu) \geqq \lambda(A,\mu) - 3\delta.$$

(II) $$\overline{\lim_{l\to\infty}}\, \lambda(A_l,\mu) \leqq \lambda(A,\mu) - 2\delta.$$

By Lemma 1.1 there is a sequence $(\nu_n)_{n=1}^{\infty} \subset A$ such that $\lambda((\nu_n)_{n=1}^{\infty},\mu) = \lambda(A,\mu)$. Since $\lambda(A_l,\mu) \leqq \lambda(A,\mu) - \delta$ for all l, for every m there is an N such that $(\nu_n)_{n=N}^{\infty} \cap (\bigcup_{l=1}^{m} A_l) = \emptyset$. Hence there is a subsequence of $(\nu_n)_{n=1}^{\infty}$ such that

$$\overline{\lim_{k\to\infty}}\, \nu_{n(k)} = \nu \quad \text{and} \quad \lambda((\nu_{n(k)})_{k=1}^{\infty},\mu) = \lambda(A,\mu),$$

and the proof of the lemma is complete.

We will show by induction on k that given indices $i(0), i(1), \cdots, i(k+1)$ such that

$$|\lambda(D(i(s),\beta)^{d(i(t))},\mu) - c| < \zeta/8$$

there is a subset M of D homeomorphic to $[1,\omega^{k+1}]$ such that

(16) $$|\lambda(M(s,\beta)^{d(t)},\mu) - c| < \zeta$$

and

$$M(s,\beta)^{d(t)} \subset D(i(s),\gamma)^{d(i(t))} \quad \text{for some} \quad \gamma \in [1,\omega^{n}]^{d(i(s))}$$

for each $\beta \in [1,\omega^{k+1}]^{d(s)}$, $0 \leqq t < s \leqq k+1$, ($c = c_m$).

If $k = 0$ and $i(0) < i(1) - 1$ ($i(0) = i(1) - 1$ is the consecutive case, which we discussed earlier) consider $D(i(1),\omega^{i(1)})^{d(i(0))}$:

$$D(i(1),\omega^{i(1)})^{d(i(0))} = \bigcup_{l=1}^{\infty} D(i(1)-1,\omega^{i(1)-1}l)^{d(i(0))}.$$

By Lemma 2.3 with $A_l = D(i(1)-1,\omega^{i(1)-1}l)^{d(i(0))}$ there is a convergent sequence $(\nu_n)_{n=1}^{\infty}$ in $D(i(1),\omega^{i(1)})^{d(i(0))}$ such that

$$|\lambda((\nu_n)_{n=1}^{\infty},\mu) - \lambda(D(i(1),\omega^{i(1)})^{d(i(0))},\mu)| < 3\zeta/4.$$

Letting $M=\overline{(\nu_n)_{n=1}^{\infty}}$ we have

$$|\lambda(M(1,\omega)^{d(0)},\mu)-c|<\zeta.$$

Suppose we have proved that such an M exists if $k\leqq p-1$ and we are given $i(0),i(1),\cdots,i(p+1)$. Consider $D(i(p+1),\omega^{i(p+1)})^{d(i(p))}$. If $i(p+1)-1>i(p)$,

$$D(i(p+1),\omega^{i(p+1)})^{d(i(p))}=\bigcup_{l=1}^{\infty} D(i(p+1)-1,\omega^{i(p+1)-1}l)^{d(i(p))}.$$

Applying Lemma 2.3 we can find a sequence $(\nu_n)_{n=1}^{\infty}\subset D(i(p+1),\omega^{i(p+1)})^{d(i(p))}$ such that $|\lambda(\nu_n)_{n=1}^{\infty},\mu)-c|<\zeta$ and if $D=\{d_\alpha\}_{\alpha\leqq\omega^n}$, $\lim_{n\to\infty}\nu_n=d_{\omega^{i(p+1)}}$. Since $\nu_n\in D(i(p+1),\omega^{i(p+1)})^{d(i(p))}$, for each n, $\nu_n=d_{\alpha_n}$ for some $\alpha_n\in[1,\omega^{i(p+1)}]^{d(i(p))}$. In the simpler case when $i(p)+1=i(p+1)$, we take $d_{\alpha_n}=d_{\omega^{i(p+1)-1}n}$. Now define

$$D_n=\overline{D(i(p),\alpha_n)}=D(i(p),\alpha_n)\cup\{d_{\alpha_n}\},\qquad n=1,2,\cdots.$$

The sets D_n are disjoint and each is homeomorphic to $[1,\omega^{i(p)}]$. Note that

$$|\lambda(D_n(i(s),\beta)^{d(i(t))},u)-c|<\zeta/8,$$

$0\leqq t<s\leqq p$. Hence by induction we can find subsets M_n of D_n, M_n homeomorphic to $[1,\omega^p]$ such that $M_n^{(p)}=\{d_{\alpha_n}\}$ and satisfying (16) (with $k+1=p$).

Let $M=\overline{\bigcup_{n=1}^{\infty}M_n}$. Clearly M is homeomorphic to $[1,\omega^{p+1}]$ and if $0\leqq t<s\leqq p$ and $\beta\in[1,\omega^{p+1}]^{d(s)}$

$$|\lambda(M(s,\beta)^{d(t)},\mu)-c|<\zeta.$$

If $s=p+1$ and $\beta\in[1,\omega^{p+1}]^{d(p+1)}$, then $\beta=\omega^{p+1}$ and $M(p+1,\omega^{p+1})^{d(p)}=(d_{\alpha_n})_{n=1}^{\infty}$. Thus $|\lambda(M(p+1,\omega^{p+1})^{d(p)},\mu)-c|<\zeta$. Finally if $0\leqq t<p$,

$$\begin{aligned}&\cup\{M(t+1,\beta)^{d(t)}:\beta\in[1,\omega^{p+1}]^{d(t+1)}\}\subset M(p+1,\omega^{p+1})^{d(t)}\\&\subset D(i(p+1),\omega^{i(p+1)})^{d(i(t))}\end{aligned}$$

and hence

$$\begin{aligned}c-\zeta&<\lambda(M(t+1,\beta)^{d(t)},u)\\&\leqq\lambda(M(p+1,\omega^{p+1})^{d(t)})\\&\leqq\lambda(D(i(p+1),\omega^{i(p+1)})^{d(i(t))},\mu)\\&<c+\zeta/8.\end{aligned}$$

The construction of M is complete.

We now prove the main theorem. The argument is much the same as the first part of the proof of the previous proposition.

PROOF. Since $\eta(\varepsilon, B_{C(\Delta)}, T^*B_{X^*}) \geqq \omega$, $P_\omega(\varepsilon) = P_\omega(\varepsilon, B_{C(\Delta)}, T^*B_{X^*}) \neq \emptyset$. Let $\nu_{\omega^\omega} \in P_\omega(\varepsilon)$. For each n choose $\nu_{\omega^n} \in P_n(\varepsilon)$ such that $\nu_{\omega^n} \xrightarrow{w^*} \nu$ and such that there is a sequence $(f_{\omega^n})_{n=1}^\infty \subset B_{C(\Delta)}$ with $d(f_{\omega^n}, 0) < 1/2^n$ and $\langle \nu_{\omega^n}, f_{\omega^n} \rangle > \varepsilon - 1/2^n$. (Recall $d(\,,\,)$ is a metric for convergence in measure μ.) Now we continue just as in the proof of Proposition 2.1 to find a subset $\{\nu_\alpha\}_{\alpha \leqq \omega^\omega}$ of $T^*B_{X^*}$ such that if $\nu_{\alpha_r} \to \nu_\alpha$ then there is a sequence $(g_r) \subset B_{C(\Delta)}$, $g_r \to 0$ in measure μ, such that $\overline{\lim}_{r\to\infty} \langle \nu_{\alpha_r}, g_r \rangle \geqq \varepsilon$.

Let $\varepsilon_0 = \sup\{\varepsilon \mid \exists (\nu_{\alpha_n})_{n=1}^\infty \ni \lambda((\nu_{\alpha_n})_{n=1}^\infty, \mu) \geqq \varepsilon$ and $\nu_{\alpha_n} \in \{\nu_\alpha\}_{\alpha \leqq \omega^\omega}^{(n)}\}$ and note that the supremum is attained. Let $(\mu_{\omega^n})_{n=1}^\infty$ be a sequence for which $\lambda((\mu_{\omega^n})_{n=1}^\infty, \mu) = \varepsilon_0$ and (by passing to a subsequence) we can assume that if $\mu_{\omega^n} \in \{\nu_\alpha\}_{\alpha \leqq \omega^\omega}^{d(s_n)}$ and $\mu_{\omega^{n+1}} \in \{\nu_\alpha\}_{\alpha \leqq \omega^\omega}^{d(s_{n+1})}$ then $s_n + n + 1 \leqq s_{n+1}$. For each n choose a subset $\{\mu_\alpha\}_{\omega^{n-1} < \alpha < \omega^n} \subset \{\nu_\alpha\}_{\alpha \leqq \omega^\omega}^{(s_{n-1}+1)} - \{\nu_\alpha\}_{\alpha \leqq \omega^\omega}^{(s_n)}$ such that $\overline{\{\mu_\alpha\}_{\omega^{n-1} < \alpha < \omega^n}} = \{\mu_\alpha\}_{\omega^{n-1} < \alpha \leqq \omega^n}$. If we let $A_n = \{\mu_\alpha\}_{\omega^{n-1} < \alpha \leqq \omega^n}$, $a = \varepsilon_0 + \varepsilon/16$, and $\xi = \varepsilon/8$, then there is an n_0 such that the hypotheses of Lemma 1.5 are satisfied for $(A_n)_{n=n_0}^\infty$. Indeed, we need only verify that $\lambda(\bigcup_{n=n_0}^\infty A_n, \mu) \leqq \varepsilon_0 + \varepsilon/16$ for some n_0. If this were not the case, we could find a sequence $(\mu_{\alpha_n})_{n=1}^\infty$ such that $\mu_{\alpha_n} \in A_n$ for all n and $\lambda((\mu_{\alpha_n})_{n=1}^\infty, u) > \varepsilon_0 + \varepsilon/16$. But then $\mu_{\alpha_n} \in \{\nu_\alpha\}_{\alpha \leqq \omega^\omega}^{(n)}$ for all n and hence $\lambda((\mu_{\alpha_n})_{n=1}^\infty, \mu) \leqq \varepsilon_0$.

Thus letting $\delta = \varepsilon/16$, we get an infinite set $M \subset N$, disjoint clopen sets $(G_n)_{n \in M}$, and subsets $\{\rho_\alpha\}_{\omega^{n-1} < \alpha \leqq \omega^n} \subset \{\mu_\alpha\}_{\omega^{n-1} < \alpha \leqq \omega^n}$, $\forall n \in M$, such that $\lambda(\bigcup_{n \in M} \{\rho_{\alpha|G_n^c} : \omega^{n-1} < \alpha \leqq \omega^n\}, \mu) < 3\varepsilon/16$. Now choose an n_1 such that

$$|\rho_{\alpha|G_n^c}| \left(\bigcup_{\substack{l \in M \\ l \geqq n_1}} G_l \right) \leqq \varepsilon/4 \quad \text{for all} \quad \alpha, \quad \omega^{n-1} < \alpha \leqq \omega^n, \quad \forall n \in M.$$

We now consider a fixed $n \in M$, $n \geqq n_1$ and the measures $\{\rho_{\alpha|G_n}\}_{\omega^{n-1} < \alpha \leqq \omega^n}$. Note that for any sequence $(\alpha_r)_{r=1}^\infty$, $\lambda((\rho_{\alpha_r|G_n}), \mu) \geqq \varepsilon_0 - 3\varepsilon/16 > 3\varepsilon/4$. Consequently we are in the same situation as at (2) in the proof of Proposition 2.1 with ε replaced by $3\varepsilon/4$. Thus by that argument if n is sufficiently large there is a subspace Y_k of $C(\Delta)$, Y_k isometric to $C(\omega^k)$ such that each $y_k \in Y_k$ is supported in G_n and

$$\|y\| \leqq \frac{2}{\varepsilon} \sup_{\omega^{k-1} < \alpha \leqq \omega^k} |\langle \rho_{\alpha|G_n}, y \rangle| \quad \forall y \in Y_k.$$

Choose $n(1), n(2), \cdots$ according to the proof of Proposition 2.1 so that we have Y_k on $G_{n(k)}$ as above and consider $Y = [Y_k]_{k=1}^\infty$. If $y \in Y$,

$$\|y\| = \sup_k \|y_k\| = \|y_{k_0}\| \quad \text{for some} \quad k_0,$$

where $y = \sum_{k=1}^{\infty} y_k$, $y_k \in Y_k$. Therefore

$$\|y\| = \|y_{k_0}\| \leqq \frac{2}{\varepsilon} \sup_{\omega^{n-1}<\alpha\leqq\omega^n} |\langle \rho_{\alpha|G_n}, y_{k_0}\rangle| \quad \text{for} \quad n = n(k_0).$$

Note that

$$|\langle \rho_\alpha, y\rangle| \geqq \big|\, |\langle \rho_{\alpha|G_n}, y_{k_0}\rangle| - |\langle \rho_{\alpha|G_n^c}, y\rangle| \,\big|$$

$$\geqq |\langle \rho_{\alpha|G_n}, y_{k_0}\rangle| - \frac{\varepsilon}{4}\|y\|.$$

Hence

$$\|y\| \leqq \frac{2}{\varepsilon} \sup_{\omega^{n-1}<\alpha<\omega^n} |\langle \rho_\alpha, y\rangle| + \frac{1}{2}\|y\|$$

or

$$\|y\| \leqq \frac{4}{\varepsilon} \sup |\langle \rho_\alpha, y\rangle|.$$

Thus

$$\|T_{|Y}^{-1}\| \leqq 4/\varepsilon.$$

REMARK 1. We assumed in the proof that $K = \Delta$, the Cantor set, so that we could use clopen sets $\{G_\alpha\}_{\alpha\leqq\omega^k}$ rather than open sets in the construction of Y_k. In the general case we can find continuous functions $\{f_\alpha\}_{\alpha\leqq\omega^k}$ supported in open sets $\{G_\alpha\}_{\alpha\leqq\omega^k}$ which closely approximate $\{1_{G_\alpha}\}_{\alpha\leqq\omega^k}$, i.e., if μ_α is the measure associated with the open set G_α, we find an open set G'_α, $G'_\alpha \subset \bar{G}'_\alpha \subset G_\alpha$, with $\mu_\alpha(G'_\alpha)$ close to $\mu_\alpha(G_\alpha)$. Then we can take f_o to be a continuous extension of $1_{G'_\alpha} + 0\cdot 1_{G'^c_\alpha}$. To insure that we get an isometric copy of $C(\omega^k)$ we construct the functions f_β for β such that $G_\beta \subset G_\alpha$ so that the support of f_β is contained in G'_α. This may entail discarding a few of the G_β's. Indeed we can find an open set $G''_\alpha \subset \bar{G}''_\alpha \subset G'_\alpha$ and let g_α be an extension of $1_{G''_\alpha} + 0\cdot 1_{G'^c_\alpha}$. Then if G''_α is chosen correctly $\int g_\alpha d_{\mu_\alpha}$ is nearly $\int f_\alpha d_{\mu_\alpha}$ and we can consider those β such that $\int g_\alpha d_{\mu_\beta}$ is close to $\int g_\alpha d_{\mu_\alpha}$.

REMARK 2. If we weaken the condition that Y be isometric to $C_0(\omega^\omega)$ to require only that it be isomorphic to $C_0(\omega^\omega)$ we can obtain this result by using the fact that there is a quotient map from $C(\Delta)$ onto $C(K)$ [4], and applying the case $K = \Delta$ to the map TQ.

COROLLARY 2.4. *If X is a complemented subspace of $C[0,1]$, which contains $C(\omega^n)$ uniformly for all n, then $C(\omega^\omega)$ is isomorphic to a complemented subspace of X.*

PROOF. $\eta(1, B_{C(\omega^n)}, B_{C(\omega^n)^*}) = n$ for each n. Thus by Proposition 0.1 there is an $\varepsilon > 0$ such that $P_n(\varepsilon, B_X, B_{X^*}) \neq \emptyset$ for all n and hence

$$\eta(\varepsilon, B_X, B_{X^*}) \geqq \omega.$$

If P is the projection, $\eta(\varepsilon, B_{C[0,1]}, P^*B_{X^*}) \geqq \omega$. By Theorem 0.2 X contains a subspace isomorphic to $C(\omega^\omega)$ and thus a complemented subspace isomorphic to $C(\omega^\omega)$ by the result of [9].

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